Curriculum Alignment Project

Math Unit

Unit Details

Title: Introduction to Functions

Level: College – Introductory (Pre-Calculus)

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Introduction to Functions

A PreCalculus Unit Plan

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Part I: Unit Plan Template

**Unit Title**
Introduction to Functions

**Teacher**

**Grade Level**
Pre-Calculus / Beginning College

**Approximate Length of Unit**
3 weeks

**Unit Organizer**

How can I analyze functions to determine their properties?
How can I use function properties to better identify or predict function behavior?

**Organizer Checklist**

Does your organizer meet these criteria?

- [ ] provides relevance; the “why” for learning
- [X] standards-based
- [ ] inquiry-based
- [ ] connects to prior knowledge
Standards Addressed

Academic Expectations and Program of Studies
(The minimum content required for all students by law)

Common Core Learning Standards

Interpreting Functions F-IF

Understand the concept of a function and use function notation
1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If \( f \) is a function and \( x \) is an element of its domain, then \( f(x) \) denotes the output of \( f \) corresponding to the input \( x \). The graph of \( f \) is the graph of the equation \( y = f(x) \).
2. Use function notation, evaluate functions by substituting both numbers and expressions for their argument variables.

Interpret functions that arise in applications in terms of the context
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function \( h(n) \) gives the number of person-hours it takes to assemble \( n \) engines in a factory, then the positive integers would be an appropriate domain for the function.
6. Calculate and interpret the average rate of change of a function over a specified interval and calculate examples of difference quotients.

Analyze functions using different representations
7. Graph functions expressed symbolically and show key features of the graph: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.
   a. Graph linear and quadratic functions and show intercepts, maxima, and minima.
   b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.
   c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

Building Functions F-BF

Build a function that models a relationship between two quantities
1. Write a function that describes a relationship between two quantities.
   a. Determine an explicit expression, a recursive process, or steps for calculation from a context.
   b. Combine standard function types using arithmetic operations. For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.
   c. Compose functions. For example, if \( T(y) \) is the temperature in the atmosphere as a function of height, and \( h(t) \) is the height of a weather balloon as a function of time, then \( T(h(t)) \) is the temperature at the location of the weather balloon as a function of time.

Build new functions from existing functions
3. Combine functions by composition and arithmetic operations.
3. Identify the effect on the graph of replacing \( f(x) \) by \( f(x) + k, k f(x), f(kx), \) and \( f(x + k) \) for specific values of \( k \) (both positive and negative); find the value of \( k \) given the graphs.
Include recognizing even and odd functions from their graphs and algebraic expressions for them.
4. a. Identify domain and range of a graph and discuss how the function gives a procedure for converting a point on the x-axis.
b. Solve an equation of the form \( f(x) = c \) graphically and algebraically.

**MATHEMATICAL PRACTICES**

1 **Make sense of problems and persevere in solving them.**
Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends.

2 **Reason abstractly and quantitatively.**
Mathematically proficient students make sense of quantities and their relationships in problem situations.

4 **Model with mathematics.**
Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

6 **Attend to precision.**
Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions, cases, and can recognize and use counterexamples. They justify their conclusions,

7 **Look for and make use of structure.**
Mathematically proficient students… can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects.

**Goal / Essential Understanding**

Functions are useful mathematical tools that help us organize information and model the real world. They are used by mathematicians, scientists, and engineers to study phenomenon and predict behavior. In this unit we will study the properties of functions and learn how to build and use function models so that we may obtain precise information about the thing or process being modeled. A solid understanding of functions is essential for the study of all the STEM disciplines.
What Do Students Have To Know and Be Able To Do in Order to Meet the Targeted Standards?

Students will know:

- The definition of a function as a set of (input,output) pairs of numbers;
- that the graph of a function is the set of (input,output)pairs drawn as points in the x,y-plane;
- That a function can be defined by a formula \( f(x) = \) provided the domain for \( x \) is stated;
- That a table of values giving \( f(x) \) at specific \( x \)-values does not tell you what the values of \( f(x) \) is at other \( x \)-values;
- That a good sketch of \( y = f(x) \), namely one that shows where the graph is rising or falling, can be obtained only by using calculus techniques (wait until next semester!)

Students will be able to do:

- Given a formula for \( f(x) \), find and simplify the value of \( f(x) \) when the input is either a number or an expression;
- Convert between geometric and algebraic representations of circles, parabolas, and lines in the plane; Find many different functions whose graphs pass through (0,0), (1,1), and (2,4);
- Find a function that models the relation between real-world quantities (such as the perimeter and area of a square).
Essential/Guiding Questions

Why do the equations $y = x^2$ and $x = \sqrt{y}$ not have the same graph in the x,y-plane?
Under what circumstances will the graph of an equation be the graph of a function?
What are the sources of student error in evaluating function and other expressions in one variable?
Given a formula $y = f(x)$, what features of the graph do you need to know in order to get a good picture of the graph?
Why is a circle not the graph of a function? How can you break up the circle into two pieces, each of which is the graph of a function.

Do the essential questions:

- [X] connect to targeted standards?
- [ ] narrow the focus of the organizer?
- [ ] connect and address all targeted standards?
- [X] encourage critical thinking skills?

Summative/End of Unit Assessment

Final exam questions at CCNY and BMCC that address functions.
See Appendix A.

Does the assessment:

- [X] assess all targeted standards?
- [ ] align to Depth of Knowledge level?
- [ ] demonstrate critical thinking skills?
- [ ] demonstrate learning in different ways?
- [ ] allow for diverse needs of students?
Scoring Criteria
Develop a scoring criteria tool that will evaluate your summative/end of unit assessment.

The standard rubrics used by CCNY and BMCC math departments to award partial credit on quizzes and final exam questions relating to functions.

Questions for Consideration
___ How well do we want them to know it and be able to do it?
___ What do we want students to know and be able to do?
___ How will we know when they know it or do it well?

Entry-level Assessment
The pre-calculus survey quiz given at CCNY
Sample questions:
   Convert 3/8 to a decimal
   Find the equation of a line joining two given points
   Solve the equation $x^2 = x$

How do I…
___ find out what my students already know and are able to do?
___ find out what additional support students need to meet a given learning target?
___ form flexible groups for instruction based on what students know and are able to do?
Type of Assessments—In addition to your summative/end of unit assessment, what other assessments will you use throughout the unit (e.g., formative, summative assessments, diagnostic assessments, pre-assessment aligned with learning targets, classroom assessments, learning checks?)

<table>
<thead>
<tr>
<th>Assessment</th>
<th>Learning target aligned to assessment</th>
<th>Write F for Formative an S for Summative (may be both)</th>
<th>How Often?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Anecdotal records</td>
<td>_____</td>
<td><em><strong><strong>f _____ _____ s</strong></strong></em></td>
<td>daily</td>
</tr>
<tr>
<td>Class discussions</td>
<td>_____</td>
<td><em><strong><strong>f _____ _____ s</strong></strong></em></td>
<td>once</td>
</tr>
<tr>
<td>Conferences and interviews</td>
<td>_____</td>
<td><em><strong><strong>f _____ _____ s</strong></strong></em></td>
<td>daily</td>
</tr>
<tr>
<td>End of unit tests (including MC and OR)</td>
<td>_____</td>
<td><em><strong><strong>f _____ _____ s</strong></strong></em></td>
<td>once</td>
</tr>
<tr>
<td>Journals, learning logs</td>
<td>_____</td>
<td><em><strong><strong>f _____ _____ s</strong></strong></em></td>
<td>daily</td>
</tr>
<tr>
<td>Performance events</td>
<td>_____</td>
<td><em><strong><strong>f _____ _____ s</strong></strong></em></td>
<td>daily</td>
</tr>
<tr>
<td>Performance tasks</td>
<td>_____</td>
<td><em><strong><strong>f _____ _____ s</strong></strong></em></td>
<td>daily</td>
</tr>
<tr>
<td>Projects</td>
<td>_____</td>
<td><em><strong><strong>f _____ _____ s</strong></strong></em></td>
<td>daily</td>
</tr>
<tr>
<td>Running records</td>
<td>_____</td>
<td><em><strong><strong>f _____ _____ s</strong></strong></em></td>
<td>daily</td>
</tr>
<tr>
<td>Selected and/or constructed responses</td>
<td>_____</td>
<td><em><strong><strong>f _____ _____ s</strong></strong></em></td>
<td>daily</td>
</tr>
<tr>
<td>Self-assessment/reflection</td>
<td>_____</td>
<td><em><strong><strong>f _____ _____ s</strong></strong></em></td>
<td>daily</td>
</tr>
<tr>
<td>Student revision of assessment answers</td>
<td>_____</td>
<td><em><strong><strong>f _____ _____ s</strong></strong></em></td>
<td>daily</td>
</tr>
<tr>
<td>Student work folder</td>
<td>_____</td>
<td><em><strong><strong>f _____ _____ s</strong></strong></em></td>
<td>daily</td>
</tr>
<tr>
<td>Other: __Homework</td>
<td>_____</td>
<td><em><strong><strong>f _____ _____ s</strong></strong></em></td>
<td>daily</td>
</tr>
<tr>
<td><em><strong><strong><strong>Quizzes</strong></strong></strong></em>_______________</td>
<td>_____</td>
<td><em><strong><strong>f _____ _____ s</strong></strong></em></td>
<td>daily</td>
</tr>
</tbody>
</table>

See Appendix B
Learning Experiences

Indicate your unit learning experiences here.

Students will participate in an active lecture format. They will check their understanding through questioning, independent and group in-class assessment, quizzes and homework completion.

See Appendix C

<table>
<thead>
<tr>
<th>How do the learning experiences...</th>
</tr>
</thead>
<tbody>
<tr>
<td>___ address individual student needs?</td>
</tr>
<tr>
<td>___ consider the perspective of the learner?</td>
</tr>
<tr>
<td>___ include varied and rigorous experiences?</td>
</tr>
<tr>
<td>___ incorporate appropriate literacy strategies/skills?</td>
</tr>
<tr>
<td>___ incorporate appropriate content literacy strategies/skills?</td>
</tr>
<tr>
<td>___ connect to other content areas as appropriate?</td>
</tr>
<tr>
<td>___ integrate technology as appropriate?</td>
</tr>
</tbody>
</table>

Unit Sequencing

Lesson 1: Introduction to functions, function representations and function notation. 3-days – *Sample lesson Included*

Lesson 2: Analyzing functions 2-days

Lesson 3: Library of Classic functions 3-4-days *Sample lesson Included*

Lesson 4: Modeling with functions 1-day
Resources/Tools
List resources/materials that are needed to support student learning.


Course Blackboard Notes


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Do the resources:
- relate to the identified targeted standards?
- enhance student learning?
- allow for the diverse needs of students?
- move beyond the textbook?
- help make learning relevant to students?
- integrate technology in a meaningful way?
Reflection
After teaching the unit, reflect on the strengths and weaknesses of the lessons, activities and assessments. How can I make the unit more effective?

Reflections of Peg Dean – Mathematics BMCC

Most of the revisions made in the lesson plans related to time estimates. Keeping in mind that the only graphing most of my Pre-calculus students have seen previously has been linear graphs. While teaching these lessons, I would like to present my graphs via computer to save time and also to improve my notes that available to students on Blackboard.

I would like to have student response systems “clickers” readily available to “instant check-ins.” This would allow me and my students to quickly assess their understanding of the properties of functions.

In reviewing the Library of Functions Hand-in Homework assignment, I found that the students results were mixed, even on the questions that I felt all students should be able to answer. I would, perhaps, assign these one at a time during a class period when we are creating the library and allow students to work together to solve them while I have time to support the struggling learners.

Reflections of Bev Smith – Mathematics Education CCNY

Providing opportunities for students to check their understanding during a lesson is a worthy goal. After observing Peg Dean teach, she and I discussed ways that technology could make her instruction more efficient. Listening to the teacher’s thinking as she determines the appropriate scale for a graph and selects critical points needed to obtain an accurate sketch is an important part of the lesson, but this can become repetitive and less effective as time goes on. Careful use of technology would allow multiple graphs to be generated efficiently and allow for more time to be focused on goal of the lesson -- properties of graphs. It is also clear from the “Hand-in Homework” that students are at different levels of readiness for the Introduction to Functions Unit. Using tools that would promote efficient use of time in the class room and formative assessment during the lesson, would allow time for more teacher-to-student interaction during class and enable the instructor to differentiate instruction.

Questions for Reflection:

- What worked well and how do I know this?
- What lessons/activities do I need to revise? Why? How?
- How did my assessments (formative/summative) guide/alter my instruction?
- Should/Could I involve other teachers in this unit (cross-content connections)?
- Are there any additional resources I need to include?
- What might I do differently next time?
Appendix A
Function Unit Test

MATH 206 Pre-Calculus             Exam 1 B             Name:

You may use a scientific calculator, but no cell phones, iBooks, etc., etc. Show your
work for partial credit. Irrational answers may be left in radical form, or as a decimal
approximation correct to the nearest hundredth. Simplify your answers where possible.

Questions 1-5 refer to the following functions:

I. \( a(x) = x^3 + 2x^2 - 8x \)

II. \( b(x) = \frac{9 - x^2}{3 + x} \)

III. \( c(x) = x^2 - 3 \)

IV. \( d(x) = \begin{cases} x^3, & \text{for } x \leq 1; \\ \sqrt{x}, & \text{for } x > 1. \end{cases} \)

V. \( e(x) = \frac{-x + 6}{2} \)

VI. \( f(x) = |x - 3| \)

VII. \( g(x) = x^3 - 1 \)

VIII. \( h(x) = -x^{\frac{1}{3}} \)

1. Evaluate
   
i) \( b(2) \)
   
ii) \( d(8) \)

   
iii) \( \frac{c(x + r) - c(x)}{r} \)
2. What is the natural domain of
   i) \( a(x) \)  
   ii) \( b(x) \)  
   iii) \( d(x) \)

3. Odd, even or neither?
   i) \( e(x) \)  
   ii) \( c(x) \)  
   iii) \( h(x) \)

4. What are the zeroes of
   i) \( a(x) \)  
   ii) \( b(x) \)  
   iii) \( g(x) \)
5. For what x-value(s) is the given function increasing?
   i) e(x)          ii) c(x)          iii) f(x)

On the accompanying graph paper, graph the following: (Extra credit: where does each function have a minimum value?)

6. \[ y = -\frac{1}{x} \]

7. \[ y = |\frac{x}{3} - 1| \]

8. \[ y = -x^3 + 2 \]

9. \[ y = \sqrt{x} \]

10. EXTRA CREDIT: \[ y = d(x) \] (from page 1).
You may use a scientific calculator, but no cell phones, iBooks, etc., etc. Show your work for partial credit. Irrational answers may be left in radical form, or as a decimal approximation correct to the nearest hundredth. Simplify your answers where possible.

Questions 1-5 refer to the following functions:

I. \( a(x) = x^3 + 2x^2 - 8x \)

II. \( b(x) = \frac{9 - x^2}{3 + x} \)

III. \( c(x) = x^2 - 3 \)

IV. \( d(x) = \begin{cases} x^3, & \text{for } x \leq 1; \\ \sqrt{x}, & \text{for } x > 1. \end{cases} \)

V. \( e(x) = \frac{-x + 6}{2} \)

VI. \( f(x) = |x - 3| \)

VII. \( g(x) = x^3 - 1 \)

VIII. \( h(x) = -x^\frac{1}{3} \)

1. Evaluate

   i) \( b(2) \)

   ii) \( d(8) \)

   iii) \( \frac{c(3 + r) - c(3)}{r} \)
Pre-Calculus: Hand-in Homework Assignment

Just as we did in class for the functions we put in our “Library of Classic Functions,” answer the following questions for the functions listed below, on a separate piece of paper. (You may find it easier to try to graph the function before you have answered all other questions; or you may want to try to answer all the questions in order to help you draw a reasonable graph.)

a) If the domain is not given what is the natural domain of the function?
b) What is the range of the function?
c) Is the function one-to-one?
d) What are the zeros of the function?
e) For what x-values is the function positive?
f) For what x-values is the function decreasing?
g) Is the function even, odd, or neither?
h) On graph paper, sketch a graph of \( y = f(x) \).

For each answer explain why you think it is correct.

1. \( f(x) = 2^x \)

2. \( f(x) = \frac{|x|}{x} \)

3. The remainder (modulo 4) function with domain \( \mathbb{Z}^+ \). Given an input \( x \), the output \( f(x) \) is the integer remainder when \( x \) is divided by 4.
Function Unit – Lessons 1, 2, & 3

Lesson Overview

<table>
<thead>
<tr>
<th>Topic Introduction to Functions</th>
<th>Time</th>
<th>3 – 1.25 Hour Classes</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Objectives:</strong> Students will define a function and identify various relations as functions or non-functions. Students will express functions using function notation. Students will substitute a value or an expression for the argument of function and represent the result in simplest form.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Check-in</td>
<td>Rewriting expressions in one variable as a simplified sum of terms</td>
<td>CCNY final exam questions on this subject</td>
</tr>
<tr>
<td>16. Given $f(x) = 1 - x^2$ and $g(x) = 2x^2$ find $f(g(-3)) - g(f(1))$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17. Let $f(x) = x - x^2 - 1$. Rewrite $\frac{f(x+h) - f(x)}{h}$ as a simplified sum.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20. Sketch the parabola $y = x^2 + 4x - 4$. Show and label the coordinates of the vertex and intercepts.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Homework: Stewart Redlin Watson question list.</td>
<td>Key Vocabulary:</td>
<td></td>
</tr>
<tr>
<td>input, output, value, function, domain, range, table, list, graph, vertical line test, substitution</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

By the end of this lesson students will be able to answer the following questions:

1. What is a function (of one variable)?
2. How are functions used to model physical problems?
3. In a physics problem that describes motion, what real-life quantity is the input to the function?
4. How are simple functions (those with finitely many inputs) represented?
5. Why do lists and tables not supply enough information to describe completely a real-life function?
6. In most situations a function is defined by a formula involving the input variable. This is not enough. What else is needed to define the function?
7. If $f(x)$ is a function with domain $[0,4]$, how much do you know (or don’t you know) about the graph of the function if you know just the values $f(0), f(1), f(2), f(3),$ and $f(4)$?
8. How do you decide whether a given graph in the x,y-plane is the graph of a function?
9. What are the domain and range of a function?
10. If a function of $f(x)$ is given as a formula, what crucial rule involving parentheses must be followed if you want to correctly find function values $f(3), f(-17), f(a), f(x + h),$ and $f(x + h) - f(x)$?
<table>
<thead>
<tr>
<th>Structure</th>
<th>Time</th>
<th>Main Ideas/Questions</th>
<th>Presentation (Board/Slideshow/Handouts)</th>
</tr>
</thead>
</table>
| Lecture   | 5    | A function is like a simple computer that works as follows: you type in a number (called an input), and the computer responds by printing out a number (called an output). You can type in as many numbers as you wish: for each number you type, the machine prints a number in response. The machine is consistent in the following sense: if you type in the input 3 on two different days, the computer will print out the same output value on both occasions. In other words, for each input, the computer types a unique output.  

We want to talk about the function and understand what the machine is doing.  

First, give the machine a name. Frank would be nice, but we generally use shorter names that consist of a single letter. Suppose a machine named $f$ prints out 4 when you type in 3.  
- We write: $f(3) = 4$;  
- we say “$f$ of 3 equals 4;”  
- we also say “the value of $f$ at 3 is 4.”  

Warning: If $f$ is a variable, $f(3)$ means $f$ times 3. But if $f$ is a function, then $f(3)$ means something completely different, namely the value of the function $f$ at input 3.  

You need to decide from context whether $f$ is a function or a variable! |
| Input 3--> | 4 Output |
| Introduction to Functions |

We name a function with a single letter, such as $f$.  
- We write: $f(3) = 4$;  
- we say “$f$ of 3 equals 4;”  
- we also say “the value of $f$ at 3 is 4.”  

At some point, you decide to quit. When you do, the numbers that you have typed in are called the domain of the function, while the numbers the machine printed out are the range of the function.  

The function description presented above is an attempt to give an everyday picture of a sometimes tricky mathematical situation.  

Some books say that a function is a rule that assigns to each number in one set a unique number in another set. However, in practice, pre-calculus and calculus study functions for which those input and output sets are sets of real numbers.  

A function is a collection of number pairs $(x, f(x))$. Each pair corresponds to a point in the $x,y$-plane.  

**Definition:** The graph of a function consists of all points $(x, f(x))$, where $x$ is in the domain of the function.
Furthermore, the ‘rule’ can be specified in many ways. Sometimes it’s important to understand the rule (what actually went on in the machine) and sometimes the rule doesn’t matter. In the final analysis, however, we think of the function simply as the collection of all number pairs \((x, f(x))\), where \(x\) is the input and \(f(x)\) is the output.

Finally, we get the graph (a picture of the function) by representing each number pair \((x,f(x))\) as a point in the plane and then plotting all such points.

To the right are official definitions of the terms we have used.

**One way to define a function is by stating a formula and specifying the domain.**

Let \(f(x) = x^2 - 1\)

If you type in \(x = 1.5\), then the value of \(f\) at 1.5 is \(f(1.5) = 1.5^2 - 1 = 0.25\). In other words, the output is 0.25 when the input is 1.5. This single fact doesn’t really tell you very much about the function. You would probably want to try other inputs as well. But how would you choose these inputs in order to get the ‘big picture’ of the function? That’s a very tricky question, which we will discuss later.

A function is a set of pairs \((x,y)\) where
- \(x\) (the input) and \(y\) (the output) are real numbers
- no two pairs have the same input

The domain of a function is all its inputs. The range of a function is all of its outputs.

**Example of a function defined by a formula:** Let \(f(x) = x^2 - 1\), where \(x\) is in the domain \([-50,50]\)

For \(x = 1.5\), \(f(1.5) = (1.5)^2 - 1 = 0.25\)

For the time being, make sure you understand that picking only whole numbers as inputs is not a good idea. To be safe, you need to plot points \((x,f(x))\) with closely spaced \(x\)-values. That’s exactly what a graphing calculator does. It’s nice to look at the graph, but relying on the calculator won’t help you understand what’s really going on. To understand why the graph looks like the calculator display, we need to use algebra (this semester, in pre-calculus) and the theory of limits (in calculus, next semester).

**WARNING:** For MOST functions, the domain does NOT consist of just whole numbers, and the graph is NOT just a collection of a finite number of dots. In many cases, you need to find the value \(f(x)\) of the function not just at \(x = 0, 1, 2, 3\), but also at in-between values of \(x\).
Representing Functions

Your text says that there are four ways to represent functions:

- **algebraically**, with a formula \( f(x) = x^2 \) or something similar;
- **graphically**, by drawing the graph
- **numerically**, by writing down a table or list of (input, output) number pairs; or
- **verbally**, by describing a real-life situation where one physical quantity (say the height of a ball above the ground) depends on another quantity (such as time).

*For any given function, some representations are more useful than others.*

2. **Represent functions**

- **algebraically**
- **graphically**
- **numerically**
- **verbally**

However, in college math, we almost never use all four representations for the same function! In fact, it’s important to be able to handle the algebra of functions without having any idea of what the function might (or might not) represent in “real life.”

### Choosing a useful representation for a function.

**Example:** We can represent a function as a list of number pairs or as a table. This works only if there are finitely many such number pairs. At the right is an example for a function with five input values 0,1,2,3,4.

#### Limitations of numerical representation

If the domain of a function is not finite (this is usually the case for functions that arise in science) we are unlikely to get a decent understanding of the function by using a small table. We can use a computer to calculate a large table.

If the domain of the function is a finite interval such as \([-10,10]\), we could use lots of very closely spaced \(x\) values in that interval to get points on the graph. Usually, but not always, you will get an accurate graph by connecting those points with line segments.

However, if the domain of the function is all real numbers, then we would need to use algebra and/or calculus to get a good description of the graph of the function.

### Numerical Representation:

**List representation for function** \( f : \)

\( f = \{(0,0),(1,1),(2,4),(3,9),(4,16)\} \)

The **domain** of \( f \) is \( \{0,1,2,3,4\} \).

The **range** of \( f \) is \( \{0,1,4,9,16\} \).

**Table representation for function** \( f : \)

<table>
<thead>
<tr>
<th>input x</th>
<th>output y</th>
<th>pair ((x,y))</th>
<th>( y = f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>((0,0))</td>
<td>0 = f(0)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>((1,1))</td>
<td>1 = f(1)</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>((2,4))</td>
<td>4 = f(2)</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>((3,9))</td>
<td>9 = f(3)</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>((4,16))</td>
<td>16 = f(4)</td>
</tr>
</tbody>
</table>

The **domain** of \( f \) is \( \{0,1,2,3,4\} \).

The **range** of \( f \) is \( \{0,1,4,9,16\} \).
Verbal representations of functions help us do science.

Suppose you want to understand the motion of a ball dropped from a 100-foot building. You could say that the ball’s height at time $t$ is given by the function $h(t)$, but this is not useful unless you have a formula for $h(t)$. Fortunately, Newtonian physics gives a precise answer: if you drop a ball from the top of a 100-foot building, then its height at time $t$ is given by the function $h(t) = 100 - 16t^2$.

By drawing the graph of the function, you can get a good picture of the ball’s motion.

In the above discussion, the sequence of representations is as follows

1. *Verbal description* of a real-life situation: drop a ball from a building.
2. Convert this to an *algebraic representation* for the function, in this case $h(t) = 100 - 16t^2$.
3. For an overall picture of the ball’s motion, draw the *graph* of the function $h(t)$ using a suitable domain. There are two ways to do this

   You can use a computer to calculate a *table* of (input, output) pairs, and then draw the graph by plotting corresponding points. This gives only a limited understanding of what is going on. It’s nice to see that there is a hilltop at a certain point on the graph, but the computer can’t explain why it’s there.

   Much better: use algebra and calculus to help you understand how the algebraic representation gives rise to the wiggles and other features in the graph.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$h(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>-44</td>
</tr>
<tr>
<td>-2</td>
<td>36</td>
</tr>
<tr>
<td>-1</td>
<td>84</td>
</tr>
<tr>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>1</td>
<td>84</td>
</tr>
<tr>
<td>2</td>
<td>36</td>
</tr>
</tbody>
</table>

If you drop a ball from the top of a 100-foot building what is the ball height at time $t$?

Use the formula $h(t) = 100 - 16t^2$, where $t$ is time and $h(t)$ is the height of the ball.

In this math class, the two most important representations of a function are algebraic and graphical.
As described above, we can move from the algebraic representation \( h(t) = 100 - 16t^2 \) to a graphical representation.

However, there are situations where you start with a curve in the x,y-plane and want to know: is what you see the graph of a function? The answer is easy to state but a bit tricky to apply:

**The vertical line test is used to decide whether a set of points in the plane is the graph of a function**

**The vertical line test states:** A set of points in the plane is the graph of a function if and only if no vertical line meets the set more than once.

Another way to say the same thing: no two distinct points have the same x-coordinate.

If a graph does pass the vertical line test, then the following is true. For any x in the domain of the function, \( f(x) \) is the y-coordinate of the unique point that lies on the vertical line through the point \((x,0)\).

*The table at the right gives a function that doesn't pass the vertical line test. Reason: there are two number pairs with the same input, 4, but different outputs, 2 and −2*

<table>
<thead>
<tr>
<th></th>
<th>y = f(x)</th>
<th>(x,y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>(4,2)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>(1,1)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>(0,0)</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>(9,3)</td>
</tr>
<tr>
<td>4</td>
<td>-2</td>
<td>(4,-2)</td>
</tr>
</tbody>
</table>

**Example 2:** A circle in the plane isn’t the graph of a function. Use the vertical line test to show why this statement is true.

To understand this example better, we need to do some review.
### 3. Expressions, Equations, and Graphs

Many sets in the plane are not graphs of functions. However, they are very often the graphs of equations. Let’s review some basic vocabulary.

**An equation in** \( x, y \) **is** a statement that two expressions in \( x, y \) are equal.

A **solution of an equation in** \( x, y \) **is** a number pair \( (a, b) \) that makes the equation true when you substitute \( a \) for \( x \) and \( b \) for \( y \). We write a solution as \( (x, y) = (a, b) \) or as \( x = a; y = b \).

The **graph of an equation in** \( x, y \) **consists** of all of its solutions, plotted as points in the \( x, y \)-plane.

---

### An equation in \( x, y \) is a statement that two expressions in \( x, y \) are equal.

A **solution of an equation in** \( x, y \) **is** a number pair \( (a, b) \) that makes the equation true.

**Example:**  
\( (x, y) = (3, 2) \) is a solution of the equation \( x + y = 5 \).

The **graph of an equation in** \( x, y \) **consists** of all of its solutions, plotted as points in the \( x, y \)-plane.

---

Next we need to review substituting a number for a letter in an expression. You know the basic idea: Figure out the expression’s value after you plug in numbers for \( x \) and \( y \).

**Example:** Is \( (x, y) = (1, -1) \) a solution of the equation \( (x^2 + y^2)(x + 7y) = 8 \)? Another way to ask the same question: Is point \( (1, -1) \) on the graph of \( (x^2 + y^2)(x + 7y) = 8 \)?

**Method:** You need to know whether the equation is true when \( x = 1 \) and \( y = -1 \). In other words, does \( (1)^2 + (-1)^2)(1 + 7(-1)) = 8 \)? Since this equation says \( -12 = 8 \), the answer is No.
It’s easy to find equations whose graph is not the graph of a function. All you need to do is to find an equation whose solution for y involves more than one formula involving x.

**Example:** is the graph of $x = y^2$ the graph of a function?

**Solution:** You have to be careful here. Try an example. If $x = -1$, then the equation $x = y^2$ becomes $-1 = y^2$, which has no solution. That means the graph of the vertical line $x = -1$ never meets the graph of the equation $x = y^2$. That’s OK: not every vertical line needs to meet the graph of a function.

However, if you let $x = 1$, then $x = y^2$ becomes $1 = y^2$, which has TWO solutions, namely $y = 1$ and $y = -1$.

**Check in**

Student independent practice and whole class review

**Exercise:** Find three vertical lines that meet the graph of the equation $x^2 + y^2 = 4$ more than once. At what points does each line meet the graph? Is the graph of the equation the graph of a function? Explain carefully.

**Why do we care about functions?**

4. Why do we care about functions?

When Isaac Newton invented calculus, he used it to explain what he called “the system of the world,” by which he meant the rules that govern and explain why the planets and stars follow the mysterious paths that we observe. In second semester calculus, you may learn how he demonstrated that the path of a planet around the sun is an ellipse. For now, you will need to be satisfied with a simpler example.

**Isaac Newton: Motion of a Ball**

If a ball is thrown upward from ground level and then released with a velocity of 64 feet per second, then Newton showed that the height of the ball at time $t$ is (approximately) $s(t) = 64t - 16t^2$ feet.

**Example 1:** At what time does the ball return to the ground?
Newton’s starting point was to say that the position of an object is a function of time. If we throw a ball up from the ground, its height above the ground is described by a function (we will call) \( s(t) \), which is the ball’s height above the ground at time \( t \). By analyzing the force exerted by gravity, Newton figured out an algebraic description of that function.

**Example 1:** At what time does the ball return to the ground?

**Method:** Solve the equation
\[
s(t) = 64t - 16t^2 = 0
\]
to see that the ball is at the ground level at \( t = 0 \) (when the problem started) and at time \( t = 4 \).

**Conclusion:** The ball is at the ground level at \( t = 0 \) (when the ball is released) and at time \( t = 4 \) (when the ball returns to the ground).

**Answer 1:** The ball returns to the ground after 4 seconds.

**Example 2:** At what time is the ball 128 feet above the ground?

**Method:** Solve the harder equation
\[
s(t) = 128
\]
to figure out when the ball is 128 feet above the ground.

\[
s(t) = 64t - 16t^2 = 128
\]
\[
4t - t^2 = 8
\]
\[
t^2 - 4t + 8 = 0
\]
\[
t = \frac{4 \pm \sqrt{16 - 32}}{2}
\]
\[
t = \frac{4 \pm \sqrt{-16}}{2} = 2 \pm 2i
\]

Since \( 2i \) is an imaginary number, the equation has no solution.

**Answer 2:** The ball is never 128 feet above the ground.

Why is that? It would have been better to ask a different sort of question, as follows:

**Example 2:** At what time is the ball 128 feet above the ground?

**Method:**
\[
s(t) = 128
\]
\[
64t - 16t^2 = 128
\]
\[
4t - t^2 = 8
\]
\[
t^2 - 4t + 8 = 0
\]
\[
t = \frac{4 \pm \sqrt{16 - 32}}{2}
\]
\[
t = \frac{4 \pm \sqrt{-16}}{2} = 2 \pm 2i
\]

**Answer 2:** The ball is never 128 feet above the ground. Read further to see why.
Q3: How high does the ball go?

**Method:** Draw the graph and try to identify the highest point on the graph. A calculator or computer makes this task easy. However, it will provide only an approximate answer.

To get the exact answer, you need to work with the rule, or formula, that defined the function. Analyzing a function’s formula to find high points and low points on its graph is a major topic in first semester calculus. However, this particular problem can be solved without calculus, by finding the coordinates of the vertex of the parabola \( s(t) = 64t - 16t^2 \).

Example 3: How high does the ball go?
From the graph below, it appears that the ball reaches 64 feet high. In fact, it’s easy to see by using either algebra or calculus that the maximum height of the ball is \( s = 64 \) feet, at time \( t = 2 \) seconds.

---

<table>
<thead>
<tr>
<th>Examples of graphs of functions</th>
<th>5. Examples of graphs of functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>You may know, and we will later review, methods for graphing several types of equations.</td>
<td></td>
</tr>
<tr>
<td>- The graph of the equation ( y = 7 ) is the horizontal line through point ((0, 7)). Why?</td>
<td></td>
</tr>
<tr>
<td>- The graph of the equation ( x = 8 ) is the vertical line through point ((8, 0)). Why?</td>
<td></td>
</tr>
<tr>
<td>- The graph of the equation ( y = 4x + 7 ) is the straight line drawn through points ((x, 4x + 7)) obtained by choosing any two values for ( x ). For example, choosing ( x = 0 ) gives point ((0, 7)), the ( y )-intercept of the graph. Choose any other ( x )-value, say ( x = 1 ), to find point ((1, 11)). The graph is obtained by drawing the straight line through the points ((0, 7)) and ((1, 11)).</td>
<td></td>
</tr>
<tr>
<td>- The graphs of the three equations ( y = x ), ( y = x^2 ), ( y = x^3 ) are plotted at the right with domain ([-1.25, 1.25])</td>
<td></td>
</tr>
</tbody>
</table>
In principle, one can obtain the graph of a function \( y = f(x) \) by choosing closely spaced x-values in the domain of \( f \) and plotting all the points \((x, f(x))\). That’s what computers do. In contrast, people use their brains. You’ll learn how to do that in calculus.

**The domain of function \( f \) is the set of all its inputs. The range of function \( f \) is the set of all its outputs.**

### Check-in

Have students work individually or in small groups:

**Exercise:** Define a function with domain \( \{1,2,3,4\} \) and range \( \{5,7\} \). How many points are on the graph of this function?

**Exercise:** Draw a careful graph of the function defined by \( f(x) = x - x^3 \) for \( x \) between -3 and 3.

### Functions with infinite domain

Functions with infinite domain are usually defined by a formula of the form

\[ f(x) = \text{an expression in } x \]

The formula isn’t enough: You must also state explicitly the domain for \( f \).

Functions with the same formula but different domains are different functions and have different graphs. Here are three related examples.

<table>
<thead>
<tr>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = 2 ); domain ( {0,1,2,3,4} )</td>
</tr>
<tr>
<td>( g(x) = x^2 ); domain ( {0,1,2,3,4} )</td>
</tr>
<tr>
<td>( h(x) = x^2 ); domain ( [0,4] ) (the interval ( 0 \leq x \leq 4 ))</td>
</tr>
</tbody>
</table>

### Check-in

Have student work individually or in small groups:

**Exercises:**

1. Find the range of each of the functions \( f, g, \) and \( h \) defined above.
2. Explain clearly why function \( h(x) \) can’t be specified by a table.
3. Graph each of the functions \( f, g, \) and \( h \).
4. Let \( G(x) = x^2 + (x-1)(x-2)(x-3)(x-4) \) with domain \( [0,4] \). Calculate \( G(x) \) for \( x = 1, 2, 3, 4 \).

State and explain the relationship between the functions \( f, g, h, \) and \( G \).

To help you think about Exercise 4, it’s useful to graph the three functions, as follows. Focus on the domain of each function.
The graph of \( g(x) = x^2 \), with domain \( \{0,1,2,3,4\} \), consists of five blue dots.

The graph of \( h(x) = x^2 \), with domain \([0,4]\), is the red curve below. It passes through the five blue dots.

The graph of \( G \), the green curve below, passes through the same five blue dots. It has the same domain as \( g \), but its graph is a complicated wiggly curve.

**Note:** In pre-calculus and calculus, an interval of real numbers (for example \([0,4]\)) is the domain of the functions that we study. The graphs above show an important principle: choosing only whole number \( x \)-coordinates may give you a very poor picture of what a graph looks like.

<table>
<thead>
<tr>
<th>Substituting in expressions and equations</th>
<th>Substituting in expressions and equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>In pre-calculus and calculus, the following sort of algebra problem is very important:</td>
<td>Let ( f(x) = 3x - x^2 ).</td>
</tr>
<tr>
<td>Suppose that function ( f ) is defined by ( f(x) = 3x - x^2 ). Find and simplify each of the following: a) ( f(3) - f(-4) ) and b) ( \frac{f(x + h) - f(x - h)}{h} )</td>
<td>Find and simplify each of the following: a) ( f(3) - f(-4) ) and b) ( \frac{f(x + h) - f(x - h)}{h} )</td>
</tr>
<tr>
<td>Answering such questions correctly and consistently is one important component of success in these courses. To do so, you need to</td>
<td>Always use parentheses when you...</td>
</tr>
</tbody>
</table>
remember just one thing:
Always use parentheses when you substitute in expressions!

In case you didn’t hear the first time:
Always use parentheses when you substitute in expressions!

Before proceeding, we need to define the word *substitute* carefully. Here’s what we mean:

To *substitute* *blah* for letter *x* in an expression, **erase each letter** *x* in the expression, then **replace** it by *(blah)*.

The parentheses are crucial. Omitting them is usually a fatal error.

Example: Substitute \(-3\) for *x* in \(3 - x^2\) and then simplify the result.
Right: \(3 - (-3)^2 = 3 - 9 = -6\)
Wrong: \(3 - (-3)^2 = 3 - 9 = 12\)

Example: Substitute *blah* (a product of four variables) for *x* in \(3 - x^2\) and simplify the result.
Right: \(3 - (blah)^2 = 3 - b^2l^2a^2h^2\)
Wrong: \(3 - blah^2\)

Example: Substitute \(x + h\) for *x* in the expression \(x^2 - 3x - 7\) and simplify the result.
Right: \((x + h)^2 - 3(x + h) - 7 \Rightarrow x^2 + 2xh + h^2 - 3x - 3h - 7\)
Wrong: \(x^2 + h - 3x + h - 7\)

Always use parentheses when you substitute in expressions!

In case you didn’t hear the third time:
Always use parentheses when you substitute in expressions!

More Examples

1. Example: Substitute \(a - 2\) for \(y\) and \(a + 3\) for \(z\) in the expression \(3z - yz\).
Method: Remember to use parentheses.
Answer: \(3(a + 3) - (a - 2)(a + 3)\)
2. Example: Rewrite the answer to the above as a simplified sum.

**Method:** Basic algebra principle:
Each time you multiply or divide expressions, you must place parentheses around the result before proceeding further. Thus

\[3(a + 3) - (a - 2)(a + 3)\]
\[= (3a + 9) - (a^2 + a - 6)\]
\[= 3a + 9 - a^2 - a + 6\]
\[= 2a + 15 - a^2\]

3. Example: Let \( f(x) = x + 3 \). Find \( 3x - f(x) \) and rewrite your answer as a simplified sum.

**Method:** Since we are substituting \( x + 3 \) for \( f(x) \), you need to place parentheses around the expression \( x + 3 \)

\[3x - f(x)\]
\[= 3x - (x + 3)\]
\[= 3x - x - 3\]
\[= 2x - 3\]

Check-in

Have students work in small groups or individually:

4. Exercise: Substitute each of the following for \( x \) in the expression \( 3 - x - x^2 \), and rewrite each answer as a simplified sum.
   a) 4
   b) \( q \)
   c) \( x + h \)

   What is the rule for substituting in expressions?

Summary Questions:

1. What is a function (of one variable)?
2. How are functions used to model physical problems?
3. In a physics problem that describes motion, what real-life quantity is the input to the function?
4. How are simple functions represented (those with finitely many inputs) represented?
5. Why do lists and tables not supply enough information to describe completely a real-life function?
6. In most situations a function is defined a formula involving the input variable. This is not enough. What else is needed to define the function?
|   | 7. If \( f(x) \) is a function with domain \([0,4]\), how much do you know (or don’t you know) about the graph of the function if you know just the values \( f(0), f(1), f(2), f(3), \) and \( f(4) \)?
|   | 8. How do you decide whether a given graph in the \( x,y \)-plane is the graph of a function?
|   | 9. **What is the rule for substituting in expressions?** |
Lesson Structure | Time | Main Ideas, questions | Board/Slideshow/Handout
---|---|---|---
Intro | 2min | best to have an idea already of what the graph is going to look like. | Every time we learn a new “type” of function, it is valuable to have a full understanding of the graph of a prototype of this type of function.

$f(x) = k$ | 10 min | natural domain; range; unless $k = 0$, no zeroes. (Let the class say what the graph looks like.) Reiterate: every coordinate pair has the form $(x, 2)$. This function is so easy, it can be difficult. | The constant function, $f(x) = k$. No matter what the input is, the output is always fixed at a given number. Let’s take $k = 2$ as a prototype.

$f(x) = x$ | 10min | Elicit from the class: Domain, range, zeroes. Every coordinate pair has the form $(x, x)$. The main feature of this function is that $x$ and $y$ are the same; different scales on the $x$- and $y$-axes would make it much harder to read this from the graph. | The identity function, $f(x) = x$.

Notice the following, that we can see very nicely from the graph:
- Whatever scale we use, the shape of the graph is the same.
- Because the constant is 2, the function is positive for all $x$-values. What happens if we choose the constant to be $-3$?
- The constant function is neither increasing nor decreasing.
- The constant function is a linear function, with a slope of 0.
- The constant function is an even function, since its graph has symmetry across the $y$-axis.

Notice the following from the graph:
- Whatever scale we use, the shape of the graph is the same.
- The function is positive for all $x$ greater than 0. Interval notation: $f(x)$ is positive for $x \in (0, \infty)$. Set-builder notation: $f(x)$ is positive for $x \in \{x \mid x > 0\}$.
- The identity function is increasing for all $x$-values.
- The identity function is a linear function, with a slope of 1.
- The identity function is an odd function, since its graph has symmetry about the origin.
Lesson Structure | Time | Main Ideas, questions | Board/Slideshow/Handout
--- | --- | --- | ---
$f(x) = x^2$ | 12 min | Elicit these facts from the class: Domain, range, zeroes. Every coordinate pair has the form $(x, x^2)$. | The squaring function, $f(x) = x^2$

Have the class write where $x$ is positive using interval notation, then using set-builder notation; ask students to choose which they prefer in this case.

Notice the following:

- Scale matters: whatever scale we choose, the shape of the graph will change. If we want to show the parabolic shape, we should either use values for $x$ fairly close to zero, or adjust the scale on the $y$-axis to keep the “cup” shape rather than a “pencil” or “needle.”

- Since $y$ is never negative, there is no point in including any part of the rectangular coordinate system below the $x$-axis. To keep the “cup” shape, it makes sense to use $x$-values more or less equally from both sides of the $y$-axis.

- The function is positive for all $x$ except $x = 0$.

- The function is decreasing for $x < 0$ and increasing for $x > 0$. Visually, from left to right, the graph goes from up to down (at the origin), then turns and goes up.

- The squaring function is an even function, since its graph has symmetry across the $y$-axis.

---

The cubing function, $f(x) = x^3$

Notice the following:

- Scale matters: whatever scale we choose, the shape of the graph will change. Close observation shows that we have adjusted the scale on the $y$-axis to give an idea of the shape without having to restrict the $x$ “window” too much.

- The function is increasing for $x < 0$ and increasing for $x > 0$. Visually, the graph goes from down to up as we view it from left to right.

- The function is an odd function, since its graph has symmetry about the origin.
<table>
<thead>
<tr>
<th>Lesson Structure</th>
<th>Time</th>
<th>Main Ideas, questions</th>
<th>Board/Slideshow/Handout</th>
</tr>
</thead>
</table>
| Other polynomials | 15 min | Class reflection/ discussion/ writing exercise  
Compare the graphs of $y = x$ and $y = x^3$ (two polynomials of odd degree). How are they the same? How do they differ? | Remarks about polynomial functions of degree higher than 1  
A second-degree polynomial will always have the shape of a parabola, with exactly one turning point, maximum or minimum depending on which way the parabola faces, up or down (show examples).  
![Graphs of y = f(x) and y = f(x) for polynomials of odd degree.](image)  
Higher degree polynomials of even degree may have more turning points, but $x^2$ is, in one important way, a model for all even degree polynomial graphs: as $x$ moves away from 0 in either direction along the x-axis, the corresponding $y$-values will either go up on both sides or go down on both sides.  
A third-degree polynomial graph could actually turn down and then turn back up (see example), but it will always have the essential shape of $y = x^3$ (or it could go the other way, from up to down).  
![Graphs of y = f(x) and y = f(x) for polynomials of odd degree.](image)  
Higher degree polynomials of odd degree, although they could have more turning points, will follow this same pattern out at the “tails”; viewed from left to right, the graph will go either from down to up or the reverse, up to down. |
<table>
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<th>Main Ideas, questions</th>
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<tbody>
<tr>
<td>$f(x) = \frac{1}{x}$</td>
<td>15min</td>
<td>Elicit from the class: Domain, range, zeroes. Plot some points, including negative $x$-values and $x$-values close to zero.</td>
<td>The reciprocal function, $f(x) = \frac{1}{x}$</td>
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Notice the following:

- This is the first discontinuous graph in the library of graphs of classic functions. The graph has 2 parts, and they can not be connected (remember that any point on a graph of a function represents an input and its output). The reciprocal function is the prototype as the graph of a rational function.
- The function is decreasing everywhere in the domain.
- The function is an *odd* function, since its graph has symmetry about the origin.
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<tr>
<td>$f(x) = \frac{</td>
<td>x</td>
<td>}{x}$</td>
<td>25 min</td>
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Class reflection/discussion/writing exercise Compare the graphs of $y = x^2$ and $y = |x|$. How are they the same? How do they differ? 

Students will have difficulty with the concept of a piecewise-defined function. Reinforce the idea of the definition of a function. (Perhaps) create a human function machine, with half the class in charge of one “piece” of the function, and the other half of the class in charge of the other “piece”. One student serves as the function “master”, directing the input to the appropriate part of the class.

- The most interesting part of the absolute value graph is the “point”, or vertex, or cusp. Typically, we should expect that the graph of any function involving absolute value will display such cusps, although this will not always be true.
- Since $y$ is never negative, there is no point in including any part of the rectangular coordinate system below the $x$-axis. To see the “V” shape clearly, it makes sense to use $x$-values more or less equally from both sides of the $y$-axis.
- The function is decreasing for $x < 0$ and increasing for $x > 0$.
- The identity function is an even function, since its graph has symmetry across the $y$-axis.
- This is not a linear function, but it is composed of 2 linear pieces. In addition to being the prototype of an absolute value function, $f(x) = |x|$ is also a classic “piecewise”-defined function, since one way to define the absolute value function is: 

$$f(x) = \begin{cases} 
  x & \text{for } x \geq 0 \\
  -x & \text{for } x < 0 
\end{cases}.$$ 

For another example of a piecewise function, see Example 3, p. 70 in the textbook.
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<td>$f(x) = \sqrt{x}$</td>
<td>15 min</td>
<td>Remind the class about rational exponent notation. Elicit from class: domain, range, zeroes. Plot a few points. The function is always increasing.</td>
<td>The square root function, $f(x) = \sqrt{x}$</td>
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<td><img src="image" alt="Square Root Function" /></td>
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What’s of interest on the graph: domain, behavior around $x = 0$.  

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<th>$f(x) = \sqrt[3]{x}$</th>
<th>15 min</th>
<th>Elicit from class: domain, range, zeroes. Plot a few points. The function is always increasing.</th>
<th>The cube root function, $f(x) = \sqrt[3]{x}$</th>
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<td><img src="image" alt="Cube Root Function" /></td>
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The step function, or greatest integer function, \( f(x) = \lfloor x \rfloor \).

Definition: \( f(x) = \lfloor x \rfloor = \text{the greatest integer less than or equal to } x \).

- The most interesting thing about this graph, of course, is the “steps.” What is the significance of the closed circle at the left end of each step, and the open circle at the right end of each step?
- Would it be incorrect to connect the steps with vertical lines?
- Is this function odd, even, or neither?
- Don’t mix up \( \lfloor x \rfloor \) with \( |x| \)!
- How weird is this? This is a function that is nowhere increasing, and yet the values of \( y \) clearly increase as we look from left to right. How can we explain this?

We graphed another function (look back and find it) that is, in fact, decreasing on every interval for which it is defined; and yet it is possible to find 2 \( x \)-values such that \( x_1 < x_2 \) and \( f(x_1) < f(x_2) \); which, in general, we expect to find only with functions that are increasing somewhere in their domain. How can we explain this?

- This is the prototype for many real-life functions, such as:

  Message units on your cell phone are calculated in minutes. If your call lasts for anywhere greater than 0 minutes but less than or equal to 1 minute, you have used one message unit; any time greater than 1 minute but less than or equal to 2 minutes counts as two message units, etc. What quantity is the input here? What quantity is the output?

  The graph of the “message unit” function is not exactly the same as the classic step function given above. Can you graph it? Can you write a rule for the function, using \( \lfloor \rfloor \) notation?